firing, separate the LM ascent stage from the descent stage, disconnect the coolant loop flow to the descent stage, rupture the ascent engine diaphragm, and force the ascent engine boundary nodes to a temperature profile consistent with a 327-sec firing. This normally corresponds to about 200 cards. Similarly, another firing would again substitute one function control card for the conventional 200.) These function control variables control events such as LM orientations, glycol coolant loop operation, initial temperatures, cabin pressurization, consumable usage, etc. This technique substituted less than 30 function control variable cards for about 2000 conventional data cards. Some function control variables are defined automatically by the TSP, others are defined manually.

Concluding Remarks

A close to real time thermal mission analysis capability was developed for the Apollo lunar landing missions. This capability will also be needed for later Apollo and Apollo Applications missions and enable quick determination of flight constraints both before and after lift-off. Turnaround time of a mission analysis typically was of the order of two months; today it can be a matter of hours.

The automated methods developed allow the use of as much automation as desired; one can perform a thermal mission analysis with over 5000 conventional data cards (the data handling subroutines in the GTA remain inactive until called) or combine a totally automated analysis (~30 cards) with conventional card input for a localized area. The automation does not replace engineering judgment, but it frees the engineer of much tedious work and gives him more time to analyze results and update the basic network. The thermal mission analysis is only as valid as the thermal network and the data contained in the various subroutines.

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A Least-Squares Estimate of the Attitude of a Satellite

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THE attitude reconstitution of a nonspin stabilized spacecraft is often reduced to finding a transformation that brings a set of measured vectors into the best coincidence with a corresponding set of model vectors. The solution of this problem for a set of n vectors ($n \geq 2$), optimal in the least squares sense, was given by Farrell and Stuelpnagel.[1] The rather lengthy computation hidden behind the general method outlined in Ref. 1 can be avoided in the special case of two noncolinear measured vectors, V_1 and V_2, and two noncolinear image vectors, W_1 and W_2, respectively. This had been pointed out by P. B. Davenport,[2] but with the condition $\|V_1\| = \|V_2\|$ and $\|W_1\| = \|W_2\|$. We show below that without this condition, a general solution is obtained by a very short algorithm. The method was developed for and applied to the attitude reconstitution of the ESRO I spacecraft, measuring the solar and magnetic vectors in the body axes.

One assumes in the following that the vectors are arranged as column vectors and the transpose of a matrix is denoted by a superscript t; to indicate the trace of a matrix, one employs the abbreviation tr. We construct a 3×2 matrix A by writing V_i in the i-th column; a similar matrix B is built by the vectors W_i. The rotation matrix R one has to deduce, transforms B into A. But the matrix $C = A - RB$ is normally not a null matrix, because of errors that may be present in B as well as in A.

The sum of all elements, raised to the square of C is a function of R and is given by

$$e(R) = tr(C^TC)$$

The matrix R, yielding the minimum of $e(R)$ is the best solution to our problem in the least squares sense. By substituting $A - RB$ for C in the preceding expression, it becomes obvious that min $e(R)$ is equivalent to

$$\max \operatorname{tr}(A^TRB)$$

because the other terms are independent of R.

To find this maximum, one introduces a rotation matrix P, which turns the vector $V_1\mathbf{X}V_2$ onto a fixed axis, and a similar rotation matrix Q, rotating $W_1\mathbf{X}W_2$ onto the same axis in the same direction. For simplicity, but without loss of generality, the z-axis is chosen for the fixed axis in the arguments following. The matrices P and Q are not unique. The matrix P can be constructed by superposing the vectors $V_1\mathbf{X}$ $(V_2\mathbf{X}V_1)$, V_1, and $V_2\mathbf{X}V_1$ in the successive rows after normalization. The matrix Q can be built in an analogous way by using W_1 and W_2. The residue is now modified as follows:

$$e'(R) = \operatorname{tr}(PA - PRQ^TQB)^T(PA - PRQ^TQB)$$

which does not affect the solution, for

$$\max \operatorname{tr}(A^TP^TPRQ^TQB) = \max \operatorname{tr}(A^TRB) \qquad (1)$$

The maximum can be computed by considering $M = PRQ^T$ as the unknown matrix, instead of R.

It remains to be proven that M is a rotation around the axis onto which the cross products of the vectors V_i and W_i are rotated. The symbol K denotes the 3×3 matrix obtained from QBA^TP^T. From the assumption for P and Q, one knows that the elements k_{i3} and k_{3i} (with $i = 1, 2, 3$) of K have to be zero.

Writing the element of M as m_{ij}, one restates the problem as

$$\begin{aligned}\max_R \operatorname{tr}(A^TRB) &= \max_M \operatorname{tr}(MK)\\ &= \max_{m_{ij}} (m_{11}k_{11} + m_{12}k_{21} + m_{21}k_{12} + m_{22}k_{22}) \qquad (2)\\ &= \max_{m_{ij}} F_0(m_{ij})\end{aligned}$$

with the constraints on the m_{ij}, which are

$$F_1 = \sum_{i=1}^{3} m_{1i}^2 - 1 = 0; \; F_2 = \sum_{i=1}^{3} m_{2i}^2 - 1 = 0 \qquad (3)$$

and

$$F_3 = \sum_{i=1}^{3} m_{1i}m_{2i} = 0 \qquad (4)$$

The given constrained extremum problem can be solved by introducing the Lagrange multipliers λ_a, λ_b and λ_c and building the expression that follows:

$$F = F_0 + \lambda_a F_1 + \lambda_b F_2 + \lambda_c F_3$$

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The six equations resulting from the computation of $(\partial F/\partial m_{ij})$, with $i = 1,2$ and $j = 1,2,3$, are

$$2\lambda_a m_{11} + \lambda_c m_{21} = -k_{11} \quad (5)$$

$$2\lambda_a m_{12} + \lambda_c m_{22} = -k_{21} \quad (6)$$

$$2\lambda_a m_{13} + \lambda_c m_{23} = 0 \quad (7)$$

$$\lambda_c m_{11} + 2\lambda_b m_{21} = -k_{12} \quad (8)$$

$$\lambda_c m_{12} + 2\lambda_b m_{22} = -k_{22} \quad (9)$$

$$\lambda_c m_{13} + 2\lambda_b m_{23} = 0 \quad (10)$$

It is not very convenient to solve the nine equations given from Eqs. (3–10) in the unknowns λ_a, λ_b, λ_c and m_{ij}. The attention can merely be focused on the value of m_{13} and m_{23}. Consider therefore the 6×6 determinant, built by the coefficients of the m_{ij} in Eqs. (5–10). Its value is obviously equal to $(4\lambda_a\lambda_b - \lambda_c^2)$,[3] and one may denote it shortly by d. As long as $d \neq 0$, it can easily be checked that $m_{13} = m_{23} = 0$ always belongs to the solution.

If d is zero, one can have $\lambda_a = \lambda_b = \lambda_c = 0$, which yields an undefined system where V_1 and V_2, or W_1 and W_2, or V_i and W_j $(i \neq j)$ are both zero vectors. Let one assume that all multipliers are different from zero and $d = 0$. In that case, the Eqs. (5) and (8), as well as Eqs. (6) and (9) become dependent. In order to be compatible, these equations yield the following condition:

$$2\lambda_a/\lambda_c = \lambda_c/2\lambda_b = k_{11}/k_{12} = k_{21}/k_{22}$$

or

$$k_{11}k_{22} - k_{12}k_{21} = 0 \quad (11)$$

If one substitutes the separate elements of PA and QB in Eq. (11), one can easily check that the condition Eq. (11) is only fulfilled if A or B contains a row of zeros, i.e., a vector V_i or W_i is a null vector. Hence, d is never zero within the scope of the problem one has to solve.

From the previous considerations, it follows that one has $m_{13} = m_{23} = m_{31} = m_{32} = 0$ and $m_{33} = \pm 1$. If $m_{33} = -1$, it is always possible to rewrite the solution for M as

$$M_- = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & -\cos\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

whereas for $m_{33} = 1$, one can state M as follows:

$$M_+ = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The angle φ has to be determined from Eq. (2). To simplify the notations, one introduces the new symbols

$$a_+ = k_{11} + k_{22};\ b_+ = k_{21} - k_{12};\ c_+ = (a_+ + b_+)^{1/2} \quad (12)$$

$$a_- = k_{11} - k_{22};\ b_- = k_{21} + k_{12};\ c_- = (a_- + b_-)^{1/2} \quad (13)$$

Now it becomes obvious that in each assumption for m_{33}, one has

$$\max_{\varphi_-} F_0(M_-) = C_-;\ \max_{\varphi_+} F_0(M_+) = C_+$$

by setting

$$\cos\varphi_\pm = a_\pm/c_\pm;\ \sin\varphi_\pm = b_\pm/c_\pm \quad (14)$$

By an explicit development of C_+ and C_-, one obtains $C_- < C_+$ because

$$a_+^2 + b_+^2 > a_-^2 + b_-^2$$

or

$$k_{11}k_{22} - k_{12}k_{21} > 0$$

which can be computed by substituting the elements of PV_i and QW_i in it. This yields

$$k_{11}k_{22} - k_{12}k_{21} = (PV_1\mathbf{X}PV_2)_z(QW_1\mathbf{X}QW_2)_z$$

which is always positive by the previous definition of P and Q. The solution is thus $R = P^TM(\varphi_+)Q$.

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A Resonance Igniter for Hydrogen-Oxygen Combustors

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One of the problems in using H_2/O_2 as propellants for a rocket, is that of devising a simple yet reliable method for multiple ignitions. Commonly used methods are high-energy spark igniters or the injection of fluorine which is hypergolic with the fuel. Both systems, however, are complex and, in the case of fluorine, there are toxicity and compatibility complications. An ignition technique that avoids these problems is based on the resonance heating,[1–3] a phenomenon which can be readily demonstrated. A hole of $\sim\frac{1}{4}$ in. diam with a depth of 1 to 3 in. is drilled into a block of wood. Air is supplied at 125 psia to a simple $\frac{1}{16}$-in.-diam choked nozzle and the nozzle is pointed into the hole. By varying the gap between the nozzle and the hole opening, a position can be found which results in the resonant coupling of the shocks in the jet with the dynamics of the gas in the hole. Intense sound is radiated, and the wood in the bottom of the hole is charred. The generally accepted model for the heating assumes that the underexpanded jet excites a periodic shock wave which oscillates up and down inside the hole, heating the gas which is trapped within the hole.

Attempts to predict the heating have been only moderately successful due to the complexity of the phenomenon. The most elaborate analysis known by the authors is that of Kang,[2] in which it was predicted that, with a perfectly insulated hole, the absolute gas temperature could be increased tenfold.

It was envisioned that a suitable igniter would be a simple choked nozzle fed by gaseous O_2 and H_2 with the nozzle mounted a fixed distance from an insulated hole (see Fig. 1). To facilitate fabrication and decrease the scope of the experiment, the following variables were arbitrarily fixed for initial testing: gas total pressure—80 psia; ambient pressure in the region between nozzle and hole—14.4 psia; nozzle diameter—0.200 in., nozzle shape—simple, choked, circular; hole diameter—0.280 in.; hole material—brass; hole-nozzle alinement—coaxial; and gas composition—stoichiometric mixture of H_2 and O_2.

The first apparatus tried consisted of a brass cylinder of 0.75-in. O.D. with an 0.25-in.-diam hole. An adjustable pis-

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